Applications of wavelets to quantum mechanics: a pedagogical example

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Abstract. We discuss in great detail two quantum mechanical models of planar electrons which are very much related to the fractional quantum Hall effect. In particular, we discuss the localization properties of the trial ground states of the models starting from considerations on the numerical results on the energy. We conclude that wavelet theory can be conveniently used in the description of the system. Finally we suggest applications of our results to the fractional quantum Hall effect.

1. Introduction

In recent years great effort has been made to find a wavefunction which minimizes the energy of a two-dimensional system of electrons subjected to a strong constant magnetic field applied perpendicularly to the sample, independently of the electron density. This is, in fact, the first step towards understanding the main features of the fractional quantum Hall effect (FQHE). Many trial ground states have been proposed so far, none of which has explained all the experimental data: the most successful is the one proposed by Laughlin, [1, 2], which describes an incompressible fluid (which therefore carries current without losing energy) whose static energy is very low.

The wavefunction proposed by Morchio, Strocchi and the present author in [3] is totally different. The authors, following the same line of reasoning [4, 5], consider the system of electrons as essentially a two-dimensional crystal. This crystal is built by first considering a Gaussian (a coherent state) centred at the origin, and then by 'moving' this Gaussian along the sites of a triangular lattice. The wavefunction of the finite volume system is the Slater determinant of the single-electron wavefunctions centred in the relevant lattice sites. The details can be found in [3] where it is also shown that, for low electron densities, the energy of this state is lower than the one obtained by Laughlin's state. However, the theoretical value of the 'critical density' at which the crystal phase appears to be favoured with respect to the liquid one is slightly different from the value given by experiment, see [6]. Therefore, even if a crystal phase is expected, its wavefunction must be refined.

In this paper we discuss a pedagogical model which suggests in which way one can modify the wavefunction in [3] to lower the energy, so as to explain the experimental data. The idea essentially consists in modifying the single-electron wavefunction trying to achieve a better electron localization. In fact, we expect that the most localized electron

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wavefunction will give the lowest value for the Coulomb energy if the electrons are originally localized around different spatial points. This claim follows both from classical and quantum considerations, see [7, 3].

The paper is organized as follows. In section 2 we introduce a physical model whose ground level is infinitely degenerate (like the one of FQHE). In this way the ground state is not fixed *a priori*. We construct different trial ground states using the Haar, the Littlewood–Paley and the harmonic oscillator bases. We also discuss their localization properties.

In section 3 we slightly modify the model previously introduced by fixing the mean positions of the electrons around lattice sites. Then we construct the new basis and we discuss how to compute the energies of the Coulomb interaction in these different bases for both the models considered.

In section 4, we give and comment on the numerical results. In the appendix, we introduce other models which can be treated with analogous techniques and, in particular, we show that the FQHE belongs to this class of model.

2. The model

Let us consider a system of N electrons in a two-dimensional device. We divide the Hamiltonian into a single-body contribution plus a two-body term:

$$H^{(N)} = \sum_{i=1}^{N} H_0(i) + \frac{1}{2} \sum_{i \neq j}^{N} \frac{1}{|r_i - r_j|}. \tag{2.1}$$

We observe that no background subtraction is considered in $H^{(N)}$. This is meaningful only if we are restricted to a finite number of electrons, that is, if we keep N to be finite, otherwise the Coulomb energy diverges when $N \to \infty$. Since we are not going to compute the true energy of the system, but only a two-body contribution, we do not need at this stage to introduce the positive background.

The form of $H_0(i)$ is chosen here in a convenient way. In fact, we are more interested in the analogies of the model in (2.1) with the FQHE than in its physical relevance. Therefore, for reasons that will appear clear in the following, we take each $H_0(i)$ to be of the form

$$H_0 = \tfrac{1}{2}(p_x^2 + x^2) + \tfrac{1}{2}p_y^2 + p_x p_y. \tag{2.2}$$

Therefore, each electron behaves like an harmonic oscillator in x, is free in y and is also subjected to a 'strange' potential which is proportional to the momentum of the electron. We also see that the z-component does not appear in $H_0(i)$ (which reflects the fact that the device is two dimensional).

It is easy to verify that the following canonical transformation

$$Q \equiv p_x + p_y \qquad P = -x$$
$$Q' \equiv p_y \qquad P' = x - y \tag{2.3}$$

preserves the commutation relations,

$$[Q, P] = [Q', P'] = i \qquad [Q, P'] = [Q, Q'] = [P, P'] = [P, Q'] = 0$$

and in this sense it is *canonical*, see [8], and that in the new variables H_0 takes the form

$$H_0 = \tfrac{1}{2}(Q^2 + P^2) \tag{2.4}$$

so that Q' and P' disappear from the definition of H_0.

To discuss the ground energy of the Hamiltonian (2.1) we follow the same steps as in [3]: first, we find the ground state of the single-electron unperturbed Hamiltonian H_0. Then,

we build up the trial ground state of the N-electrons unperturbed Hamiltonian, $\sum_i H_0(i)$, as a Slater determinant of these single-electron wavefunctions. Finally, we compute the matrix element of the Coulomb interaction in this state. As widely discussed in [3], this procedure also seems to be justified, at least for small electron densities.

Before going on, let us show the relation between our pedagogical model and the FQHE. The FQHE is described by the same Hamiltonian as in (2.1) with a different H_0,

$$H_0^F = \tfrac{1}{2}(p_x - y/2)^2 + \tfrac{1}{2}(p_y + x/2)^2$$

which describes an electron subjected to a constant magnetic field perpendicular to the device. A canonical transformation similar to the one in (2.3), see [3] and references therein, transforms H_0^F in the same H_0 (2.4). Therefore, even if H_0 in (2.2) and H_0^F are different, they are both 'projected' into the same harmonic oscillator (2.4) by different canonical transformations. This difference appears explicitly in the integral transformation rule which relates the expressions for the wavefunctions in the variables (x, y) and (Q, Q'), see [8]. The reason why we discuss this pedagogical example and not the FQHE directly is that this transformation rule is very simple for our H_0, while it is much more difficult for H_0^F, see [3]. However, a first real application of our approach in the context of FQHE can be found in [9].

From [8] we can easily find that, if $\Psi(x, y)$ and $\Phi(Q, Q')$ are respectively the wavefunctions in the ordinary space and in the 'canonically transformed' space, they are related by

$$\Psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dQ' \, \Phi(Q, Q') \exp\{i[Q'(y - x) + Qx]\}. \quad (2.5)$$

To find the single-electron ground state of our model, $\Psi_0(x, y)$, it is therefore sufficient (actually equivalent) to find the ground state $\Phi_0(Q, Q')$ of (2.4). Due to the particular form of this H_0 we see that $\Phi_0(Q, Q')$ can be factorized and that the dependence on Q is fixed: regarding the Coulomb interaction as a perturbation of H_0, it is reasonable to put

$$\Phi(Q, Q') = \frac{1}{\pi^{1/4}} \exp(-Q^2/2)\phi(Q'). \quad (2.6)$$

Here the function $\phi(Q')$ is totally free because the variable Q' does not appear in the Hamiltonian (2.4) and the energy of the unperturbed system only depends on Q. Only the form of the perturbation is fixed. An analogous phenomenon, known as the *degeneracy of the Landau levels*, takes place for H_0^F, for the same reason. This explains why the ground state of the FQHE is not fixed.

We are now ready to discuss different choices of $\phi(Q')$ and their projections in the configuration space via (2.5), comparing the resulting energies and localizations.

We start by considering the Littlewood–Paley orthonormal basis of wavelets. We refer to [10] and [11] for all that concerns the wavelet theory used in this paper.

The mother wavelet of this set is

$$L(x) = (\pi x)^{-1}(\sin(2\pi x) - \sin(\pi x))$$

which, using the well known definition $L_{mn}(x) \equiv 2^{-m/2}L(2^{-m}x - n)$, generates an orthonormal set in $L^2(\mathbb{R})$. In particular, we will be interested in the subset $\{L_m(x)\} \equiv \{L_{m0}(x)\}$. The functions of this set obviously satisfy the orthonormality condition

$$\langle L_m, L_n \rangle = \delta_{mn}$$

and, when used in (2.5) as different choices for the function $\phi(Q')$, they give the following set of wavefunctions in the configuration space:

$$\Psi_m^{(\mathrm{LP})}(x, y) = \frac{2^{m/2}}{\sqrt{2\pi}^{3/4}}\,\mathrm{e}^{-x^2/2}\chi_{D_x}(y). \tag{2.7}$$

Here $\chi_{D_x}(y)$ is the characteristic function of the set D_x, which is equal to one if $y \in D_x$ and zero otherwise. We have defined

$$D_x = \left[x - \frac{2\pi}{2^m}, x - \frac{\pi}{2^m}\right] \cup \left[x + \frac{\pi}{2^m}, x + \frac{2\pi}{2^m}\right]. \tag{2.8}$$

Due to the canonicity of the transformation (2.3) the functions of the set $\{\Psi_m^{(\mathrm{LP})}(x, y) : m \in \mathbb{Z}\}$ are obviously mutually orthonormal. Moreover, we see from (2.7) that they are very well localized in both x and y.

Another possible choice for the function $\phi(Q')$ is any function belonging to the Haar wavelet set. The mother wavelet of this set is the function

$$H(x) = \begin{cases} 1 & \text{if } 0 \leqslant x < 1/2 \\ -1 & \text{if } 1/2 \leqslant x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the relevant set is defined in the usual way: $\{H_m(x)\} \equiv \{H_{m0}(x)\} = \{2^{-m/2}H(2^{-m}x) : m \in \mathbb{Z}\}$. This set is again orthonormal but the localization of each wavefunction is rather poor. From (2.5) we get

$$\Psi_m^{(\mathrm{H})}(x, y) = \frac{2^{-m/2}\mathrm{i}}{\sqrt{2\pi}^{3/4}}\frac{\mathrm{e}^{-x^2/2}}{(y - x)}(\mathrm{e}^{\mathrm{i}2^{m-1}(y-x)} - 1)^2 \tag{2.9}$$

which again decreases exponentially in x but goes like $1/y$ in y. We do not expect therefore that the set $\{\Psi_m^{(\mathrm{H})}(x, y) : m \in \mathbb{Z}\}$ can play a relevant role in the energy computation.

We end this section by discussing another class of trial ground states of the Hamiltonian H_0. This time we will take non-wavelet functions. In particular, we consider for $\phi(Q')$ the first three eigenstates of the Hamiltonian $H_0 = \frac{1}{2}(Q'^2 + P'^2)$, which are orthonormal, and compute the projections in the configuration space of the complete function $\Phi(Q, Q')$ using the transformation rule (2.5). We easily find the following results:

$$\Psi_0^{(\mathrm{HO})}(x, y) = \frac{1}{\sqrt{\pi}}\exp[-(y^2 + 2x^2 - 2xy)/2]$$

$$\Psi_1^{(\mathrm{HO})}(x, y) = \mathrm{i}\sqrt{\frac{2}{\pi}}(y - x)\exp[-(y^2 + 2x^2 - 2xy)/2]$$

$$\Psi_2^{(\mathrm{HO})}(x, y) = \frac{1}{\sqrt{2\pi}}(1 - 2(y - x)^2)\exp[-(y^2 + 2x^2 - 2xy)/2].$$

All these wavefunctions, obviously mutually orthogonal, have a good localization in both x and y. In particular the best localized is, of course, $\Psi_0^{(\mathrm{HO})}(x, y)$.

The reason for considering these wavefunctions in this paper is that the choice of the ground state of the harmonic oscillator, which here gives $\Psi_0^{(\mathrm{HO})}(x, y)$, is the one which allows the construction of the trial ground state used in the description of the FQHE, see [3]. It is therefore useful, in our opinion, to have a comparison between these different approaches.

3. Energy computation and wavelet bases on a lattice

We start this section by discussing the way in which the energy of the system can be computed. As a matter of fact, we are not going to compute the true energy of the system, since in any case this is not physically very interesting, but only a certain matrix element which is enough to get some relevant information on the ground state of the model, since it contains the main contribution to the energy. First of all, we fix $N = 2$ in (2.1) since, in any case, the total energy is essentially a sum of two-body contributions.

In any book of many-body theory it is shown that the computation of the energy of an N-electron system, in the Hartree–Fock approximation, is a (summation of the) difference of two contributions, called, respectively, the direct and exchange terms. We should add to this difference also the ground energy of the kinetic Hamiltonian $\sum_{i=1}^{N} H_0(i)$. However, in our model, as well as in the FQHE, this contribution is constant, in the sense that it does not depend on the particular choice of the function ϕ in (2.6). Therefore it will be neglected in all future considerations.

For our two-electron system the wavefunction is the following Slater determinant:

$$\Psi(r_1, r_2) = \frac{1}{\sqrt{2!}} (\Psi_1(r_1)\Psi_2(r_2) - \Psi_2(r_1)\Psi_1(r_2)) \tag{3.1}$$

where $\Psi_i(r_j), i, j = 1, 2$, are the single-electron wavefunctions obtained in the previous section, see (2.7), (2.9) and (2.10). The Coulomb energy E_C of the system is therefore

$$E_C \equiv \int d^2r_1 \int d^2r_2 \frac{\Psi(r_1, r_2)^* \Psi(r_1, r_2)}{|r_1 - r_2|} = V_d - V_{ex}$$

where

$$V_d = \int d^2r_1 \int d^2r_2 \frac{|\Psi_1(r_1)|^2 |\Psi_2(r_2)|^2}{|r_1 - r_2|}$$

and

$$V_{ex} = \int d^2r_1 \int d^2r_2 \frac{\Psi_1(r_1)^* \Psi_2(r_2)^* \Psi_1(r_2) \Psi_2(r_1)}{|r_1 - r_2|}$$

are, respectively, the direct and exchange terms.

It is well known that, at least for localized wavefunctions, the exchange contribution is much smaller than the direct one. This feature is explicitly discussed, for instance, in [3], where these contributions are explicitly computed for the FQHE. Therefore in this paper we will not compute V_{ex}, since it is not expected to change the numerical results significantly. We will focus our attention only on the computation of the direct term V_d, which, with a little abuse of language, will still often be called the 'energy' of the system.

Before computing the expressions of V_d for the Littlewood–Paley and harmonic oscillator bases we use these same bases as starting points to introduce a 'natural' lattice into our model. The reason for doing this is again that a lattice is a natural structure for the FQHE at least for small electron densities, see [3]. Actually, since we are dealing with only two electrons, we will think of our lattice as two spatially not coincident points. We again refer to [3], and references therein, for the details concerning the construction of the lattice associated to H_0.

Using the results of the previous section it is easy to prove that the unitary operators

$$T_1 = e^{iQ'a} \qquad T_2 = e^{iP'b} \tag{3.2}$$

both commute with H_0 since they do not depend on Q and P. Moreover, they also commute with each other if $ab = 2\pi N, \forall N \in \mathbb{Z}$. From the definition (2.3), one can

also observe that, for any function $f(x, y) \in L^2(\mathbb{R})$, $T_1 f(x, y) = f(x, y + a)$ and $T_2 f(x, y) = \exp[i(x - y)b] f(x, y)$. Therefore, T_2 is simply a multiplication for a phase while T_1 acts like a shift operator. This is enough for our present aim: we can take the two different sites of our lattice along the y-axis, with a 'lattice' distance $a = 2\pi$.

In particular, defining

$$\Phi_m^{(LP)}(x, y) \equiv T_1 \Psi_m^{(LP)}(x, y) = \frac{2^{m/2}}{\sqrt{2}\pi^{3/4}} e^{-x^2/2} \chi_{D_{x-a}}(y) \tag{3.3}$$

from the Littlewood–Paley wavelets (2.7), and

$$\Phi_0^{(HO)}(x, y) \equiv T_1 \Psi_0^{(HO)}(x, y) = \frac{1}{\sqrt{\pi}} \exp[-((y + a)^2 + 2x^2 - 2x(y + a))/2] \tag{3.4}$$

for the most localized function of the harmonic oscillator states, (2.10), we conclude that both $\Phi_m^{(LP)}(x, y)$ and $\Phi_0^{(HO)}(x, y)$ are eigenstates of H_0 belonging to the ground level. This simply follows from the commutation rule $[T_1, H_0] = 0$.

It is also easy to verify that $\langle \Psi_m^{(LP)}, \Phi_m^{(LP)} \rangle = 0, \forall m \geq 1$.

The situation is a bit different for the oscillator wavefunctions; the scalar product gives $\langle \Psi_0^{(HO)}, \Phi_0^{(HO)} \rangle = e^{-\pi^2}$. This implies that the Slater determinant is normalized within an error of $e^{-2\pi^2} = O(10^{-9})$. Therefore this extra contribution can safely be neglected here, and we will work with $\Psi_0^{(HO)}$ and $\Phi_0^{(HO)}$ as if they were mutually orthogonal.

We continue this section by manipulating V_d for two different models. In the first one, which we call the 'non-lattice model', the electrons are both localized around the origin but they are described by different wavefunctions (this is necessary in order not to annihilate the Slater determinant (3.1)); and in the second, the 'lattice model', the electrons are described by the same wavefunction localized around different points in space. Of course, this is the model which is more similar to the FQHE as already discussed in [3], and in this perspective it has a particular interest.

We will omit the computation of the energy with the Haar basis $\{\Psi_m^{(H)}(x, y)\}$ since it is not expected to be relevant for understanding the FQHE. This is because the wavefunctions $\Psi_m^{(H)}(x, y)$ are the most delocalized functions within the ones we have introduced in the previous section, so that the Coulomb energy is expected to be larger than that obtained by the other bases.

3.1. Non-lattice model

We start with manipulating the expression of V_d for the basis in (2.7). From now on we will omit the index 'd' since we will be concerned only with the direct contribution to the energy. Moreover, to make explicit the dependence on the quantum numbers m, n and on the basis, we put

$$V_{LP}^{m,n} = \int d^2 r_1 \int d^2 r_2 \frac{|\Psi_m^{(LP)}(r_1)|^2 |\Psi_n^{(LP)}(r_2)|^2}{|r_1 - r_2|} \tag{3.5}$$

where $m \neq n$ because of the Pauli principle. The integration in y_1 can be easily performed. After some manipulation and change of variables we can also perform the integration in x_1 and we obtain

$$V_{LP}^{m,n} = \frac{2^{m+n}}{4\pi^3} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx \, e^{-x^2/2} \int_{x+\pi/2^n}^{x+2\pi/2^n} dt \, \log[\phi_{mn}(x, t)] \tag{3.6}$$

where we have defined the following function

$$\phi_{mn}(x, t) \equiv \frac{(t - \pi/2^m + \sqrt{x^2 + (t - \pi/2^m)^2})(t + 2\pi/2^m + \sqrt{x^2 + (t + 2\pi/2^m)^2})}{(t + \pi/2^m + \sqrt{x^2 + (t + \pi/2^m)^2})(t - 2\pi/2^m + \sqrt{x^2 + (t - 2\pi/2^m)^2})}$$

$$\times \frac{(t - \pi(3/2^n + 1/2^m) + \sqrt{x^2 + (t - \pi(3/2^n + 1/2^m))^2})}{(t - \pi(3/2^n - 1/2^m) + \sqrt{x^2 + (t - \pi(3/2^n - 1/2^m))^2})}$$

$$\times \frac{(t - \pi(3/2^n - 2/2^m) + \sqrt{x^2 + (t - \pi(3/2^n - 2/2^m))^2})}{(t - \pi(3/2^n + 2/2^m) + \sqrt{x^2 + (t - \pi(3/2^n + 2/2^m))^2})}.$$

It is possible to see that the above integral is certainly defined for all n different from $m \pm 1$. The integration can be easily performed numerically and the results are discussed in section 4.

To compute the energy for the harmonic oscillator wavefunctions (2.10) it is better to use the following equality:

$$\frac{1}{|r_1 - r_2|} = \frac{1}{2\pi} \int \frac{d^2 k}{|k|} \exp[-ik \cdot (r_1 - r_2)].$$

In this way the integrations in r_1 and r_2 in V_d are reduced to Gaussian integrals and therefore can be easily performed. Calling $V_{HO}^{i,j}$ the 'energies' related to the wavefunctions in (2.10), we find:

$$V_{HO}^{0,1} = \frac{1}{2\pi} \int \frac{d^2 k}{|k|} \left(1 - \frac{k_y^2}{2}\right) \exp[-(k_x^2 + 2k_y^2 + 2k_x k_y)/2] \tag{3.7}$$

$$V_{HO}^{0,2} = \frac{1}{4\pi} \int \frac{d^2 k}{|k|} k_y^2 \left(2 - \frac{k_y^2}{4}\right) \exp[-(k_x^2 + 2k_y^2 + 2k_x k_y)/2] \tag{3.8}$$

$$V_{HO}^{1,2} = \frac{1}{4\pi} \int \frac{d^2 k}{|k|} \left(1 - \frac{k_y^2}{2}\right) k_y^2 \left(2 - \frac{k_y^2}{4}\right) \exp[-(k_x^2 + 2k_y^2 + 2k_x k_y)/2]. \tag{3.9}$$

3.2. Lattice model

In this subsection we use the wavefunctions (3.3) and (3.4) obtained using the shift operator T_1. We start by considering the Littlewood–Paley basis.

As we have already said this time the energy is computed using the same wavefunction centred in different lattice sites. Therefore it depends only on a quantum number, m.

We call this energy

$$V_{LP}^{(m)} = \int d^2 r_1 \int d^2 r_2 \frac{|\Psi_m^{(LP)}(r_1)|^2 |\Phi_m^{(LP)}(r_2)|^2}{|r_1 - r_2|}. \tag{3.10}$$

The computation of this matrix element follows the same steps as the calculus of the analogous contribution in (3.5), and one obtains a similar expression:

$$V_{LP}^{(m)} = \frac{2^{2m}}{4\pi^3} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx \, e^{-x^2/2} \int_{x+2\pi+\pi/2^m}^{x+2\pi+2\pi/2^m} dt \, \log[\phi_m(x, t)] \tag{3.11}$$

where

$$\phi_m(x, t) \equiv \frac{(t + 2\pi/2^m + \sqrt{x^2 + (t + 2\pi/2^m)^2})(t - 4\pi/2^m + \sqrt{x^2 + (t - 4\pi/2^m)^2})}{(t + \pi/2^m + \sqrt{x^2 + (t + \pi/2^m)^2})(t - 5\pi/2^m + \sqrt{x^2 + (t - 5\pi/2^m)^2})}$$

$$\times \frac{(t - \pi/2^m + \sqrt{x^2 + (t - \pi/2^m)^2})^2}{(t - 2\pi/2^m + \sqrt{x^2 + (t - 2\pi/2^m)^2})^2}.$$

It is possible to prove that the integral surely exists for $m > 1$, which is a constraint satisfied in our conditions since we are interested in studying the behaviour of the wavefunctions and of the energy for large values of m. The reason for this interest is that for large m the wavefunctions are more localized even in the variable y, as one can see from (2.7) and (2.8).

We see that the result is very similar to that for $V_{\text{LP}}^{m,n}$, as expected. However, we will show in section 4 that the numerical outputs are very different.

We end this section by simplifying the expression of the matrix element of the Coulomb energy within the ground state of the harmonic oscillator and its translation. Using the integral formula for the Coulomb potential, we get

$$
\begin{aligned}
V_{\text{HO}}^0 &\equiv \int d^2 r_1 \int d^2 r_2 \frac{|\Psi_0^{(\text{HO})}(r_1)|^2 |\Phi_0^{(\text{HO})}(r_2)|^2}{|r_1 - r_2|} \\
&= \frac{1}{2\pi} \int \frac{d^2 k}{|k|} \exp[-2\pi i k_y - k_y^2 - k_x^2/2 - k_x k_y].
\end{aligned}
\tag{3.12}
$$

The reason why only $\Psi_0^{(\text{HO})}(r)$ is considered here is essentially that this is the most localized function among all the harmonic oscillator wavefunctions.

4. Numerical results and comments

In this section we will discuss the numerical results for both the models proposed previously and we comment on these results, paying particular attention to the localization properties of the wavefunctions.

We start by considering the non-lattice model. We report in table 1 the results for $V_{\text{LP}}^{m,n}$ for various values of (m, n). The energies of the harmonic oscillator are easily computed:

$$
V_{\text{HO}}^{0,1} = 0.918\,73 \qquad V_{\text{HO}}^{0,2} = 0.390\,19 \qquad V_{\text{HO}}^{1,2} = 0.110\,41.
\tag{4.1}
$$

For the Littlewood–Paley basis in the context of the lattice model the situation is summed up in table 2, while the energy of the harmonic oscillator is, in this case,

$$
V_{\text{HO}}^0 = 0.165\,15.
\tag{4.2}
$$

We can now comment on these results. The first obvious consideration is that, while for the first model the energy increases as much as m and n both increase, for the lattice the situation is just the opposite: the energy decreases for increasing m. Let us try to explain this different behaviour. From the definitions (2.7), (2.8) and (3.3) we see that when m increases the supports in y of the functions decrease. Therefore both $\Psi_m^{(\text{LP})}(r)$ and $\Phi_m^{(\text{LP})}(r)$ improve their localization for increasing m. Since the electrons are localized at a distance of 2π, we return to a situation similar to the one of the FQHE. It is well known that the lower bound for the ground energy is obtained if the electrons are punctually localized on the lattice sites, that is, in the classical limit. This is because in this way the distance between the electron is maximized, and therefore the Coulomb interaction obtains its minimum value. These considerations explain very well the results in table 2. We see, in fact, that when m increases the energy decreases from $0.270\,83$ to the asymptotic value $0.160\,66$. This value is already reached for $m = 9$ and stays essentially unchanged even for larger m. This value could also be predicted in an heuristic way. From the definition of $\Psi_m^{(\text{LP})}(r)$ we deduce that, for m very large, this function behaves like a $\Psi_\infty^{(\text{LP})}(r)$ whose square modulus is

$$
|\Psi_\infty^{(\text{LP})}(r)|^2 = \frac{1}{\sqrt{\pi}} e^{-x^2} \delta(x - y).
\tag{4.3}
$$

Table 1. Values of the matrix elements in (3.6) for different values of (m, n).

$V_{LP}^{(1,3)} = 0.45784$	$V_{LP}^{(1,4)} = 0.43955$	$V_{LP}^{(1,5)} = 0.43520$
$V_{LP}^{(1,6)} = 0.43412$	$V_{LP}^{(1,7)} = 0.43385$	$V_{LP}^{(1,8)} = 0.43378$
$V_{LP}^{(1,9)} = 0.43377$	$V_{LP}^{(1,m)} = 0.43376, \ m \geqslant 10$	
$V_{LP}^{(2,4)} = 0.77328$	$V_{LP}^{(2,5)} = 0.75524$	$V_{LP}^{(2,6)} = 0.75093$
$V_{LP}^{(2,7)} = 0.74986$	$V_{LP}^{(2,8)} = 0.74959$	$V_{LP}^{(2,9)} = 0.74953$
$V_{LP}^{(2,10)} = 0.74951$	$V_{LP}^{(2,m)} = 0.74950, \ m \geqslant 11$	
$V_{LP}^{(3,5)} = 1.14544$	$V_{LP}^{(3,6)} = 1.12849$	$V_{LP}^{(3,7)} = 1.12445$
$V_{LP}^{(3,8)} = 1.12345$	$V_{LP}^{(3,9)} = 1.12320$	$V_{LP}^{(3,10)} = 1.12314$
$V_{LP}^{(3,11)} = 1.12313$	$V_{LP}^{(3,m)} = 1.12312, \ m \geqslant 15$	
$V_{LP}^{(4,7)} = 1.51843$	$V_{LP}^{(4,8)} = 1.51449$	$V_{LP}^{(4,9)} = 1.51352$
$V_{LP}^{(4,10)} = 1.51327$	$V_{LP}^{(4,11)} = 1.51321$	$V_{LP}^{(4,m)} = 1.51319, \ m \geqslant 15$
$V_{LP}^{(5,8)} = 1.91054$	$V_{LP}^{(5,9)} = 1.90662$	$V_{LP}^{(5,10)} = 1.90565$
$V_{LP}^{(5,11)} = 1.90541$	$V_{LP}^{(5,m)} = 1.90533, \ m \geqslant 15$	
$V_{LP}^{(6,8)} = 2.31870$	$V_{LP}^{(6,9)} = 2.30226$	$V_{LP}^{(6,10)} = 2.29835$
$V_{LP}^{(6,11)} = 2.29738$	$V_{LP}^{(6,m)} = 2.29705, \ m \geqslant 15$	
$V_{LP}^{(7,9)} = 2.71004$	$V_{LP}^{(7,10)} = 2.69358$	$V_{LP}^{(7,11)} = 2.68966$
$V_{LP}^{(7,m)} = 2.68838, \ m \geqslant 15$		
$V_{LP}^{(8,10)} = 3.10119$	$V_{LP}^{(8,11)} = 3.08475$	$V_{LP}^{(8,15)} = 3.07956$
$V_{LP}^{(8,m)} = 3.07954, \ m \geqslant 20$		
$V_{LP}^{(9,11)} = 3.49229$	$V_{LP}^{(9,12)} = 3.47584$	$V_{LP}^{(9,13)} = 3.47192$
$V_{LP}^{(9,15)} = 3.47071$	$V_{LP}^{(9,m)} = 3.47063, \ m \geqslant 20$	
$V_{LP}^{(10,12)} = 3.88337$	$V_{LP}^{(10,13)} = 3.86691$	$V_{LP}^{(10,15)} = 3.86202$
$V_{LP}^{(10,m)} = 3.86171, \ m \geqslant 20$		
$V_{LP}^{(15,20)} = 5.81737$	$V_{LP}^{(15,m)} = 5.81705, \ m \geqslant 25$	

Table 2. Values of the matrix elements in (3.11) for different values of m.

$V_{LP}^{(1)} = 0.27083$	$V_{LP}^{(2)} = 0.17462$	$V_{LP}^{(3)} = 0.16376$
$V_{LP}^{(4)} = 0.16141$	$V_{LP}^{(5)} = 0.16085$	$V_{LP}^{(6)} = 0.16071$
$V_{LP}^{(7)} = 0.16067$	$V_{LP}^{(m)} = 0.16066, \ m > 7$	

An analogous formula holds for $\Phi_\infty^{(LP)}(r)$, $|\Phi_\infty^{(LP)}(r)|^2 = \frac{1}{\sqrt{\pi}} e^{-x^2} \delta(x - 2\pi - y)$. We can compute the energy E_∞ in this limit and we get

$$E_\infty \equiv \int d^2 r_1 \int d^2 r_2 \frac{|\Psi_\infty^{(LP)}(r_1)|^2 |\Phi_\infty^{(LP)}(r_2)|^2}{|r_1 - r_2|}$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx \, e^{-x^2/2}}{\sqrt{x^2 + 2\pi x + 2\pi^2}} = 0.16066.$$

We observe that this result exactly coincides with the one obtained on making m increase. We can conclude that the wavefunctions are really more and more localized since, in fact, their square modulus converges to an exponential function in x multiplied by $\delta(x - y)$.

We furthermore observe that the energy of the harmonic oscillator, $V_{HO}^0 = 0.16515$, is slightly larger than almost all the $V_{LP}^{(m)}$. This difference could be interpreted again as a

better localization of the functions $\Psi_m^{(LP)}(r)$ with respect to $\Psi_0^{(HO)}(r)$. This is the reason why we have only computed the energy for this wavefunction and not, say, for $\Psi_1^{(HO)}(r)$ or $\Psi_2^{(HO)}(r)$ whose localization is a bit worse. These numerical results strongly suggest the use of wavelet instead of oscillator functions for computing the energy even in the FQHE, in an attempt to better explain the phase transition between the Wigner and Laughlin phases, as discussed in the introduction.

Analogous conclusions can be obtained by considering the results in table 1. We have left these results to the end since they are less directly connected with the picture of the FQHE, since no lattice is present in this model.

We first add extra information to table 1: all the results turn out to be symmetric under the exchange $m \leftrightarrow n$, as they must.

This time the two electrons are both localized around the origin. We expect the most localized wavefunctions to have the maximum overlap between them and, therefore, the energy will be at a maximum. In a classical picture it would be as if we put two point-like charges on the same point. Of course this system is not stable and we expect very high energy for this configuration. This is exactly what happens. If we try to compute the energy $V_{LP}^{m,n}$ for m and n very large, we expect the result to be the same as that obtained by computing

$$V_{LP}^{\infty} \equiv \int d^2r_1 \int d^2r_2 \frac{|\Psi_\infty^{(LP)}(r_1)|^2|\Psi_\infty^{(LP)}(r_2)|^2}{|r_1 - r_2|} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx\, e^{-x^2/2}}{\sqrt{x^2}}$$

which diverges, as expected.

Moreover, we also see from table 1 that for each m fixed, when n increases, $V_{LP}^{m,n}$ converges toward an asymptotic value. These values could be predicted with great precision by considering the following quantities:

$$
\begin{aligned}
V_{LP}^{(m,\infty)} &\equiv \int d^2r_1 \int d^2r_2 \frac{|\Psi_m^{(LP)}(r_1)|^2|\Psi_\infty^{(LP)}(r_2)|^2}{|r_1 - r_2|} \\
&= \frac{2^{m-1}}{\pi^2}\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2} \log[\Phi_m(x)]
\end{aligned}
\tag{4.4}
$$

where we have defined

$$\Phi_m(x) = \frac{(x + 2\pi/2^m + \sqrt{x^2 + (x + 2\pi/2^m)^2})(x - \pi/2^m + \sqrt{x^2 + (x - \pi/2^m)^2})}{(x + \pi/2^m + \sqrt{x^2 + (x + \pi/2^m)^2})(x - 2\pi/2^m + \sqrt{x^2 + (x - 2\pi/2^m)^2})}.$$

The numerical results are reported in table 3. We see that these results are extremely good, in the sense that they coincide with the asymptotic values of $V_{LP}^{m,n}$, for any m fixed.

Table 3. Values of the matrix elements in (4.4) for different values of m.

$V_{LP}^{(1,\infty)} = 0.433\,76$	$V_{LP}^{(2,\infty)} = 0.749\,50$	$V_{LP}^{(3,\infty)} = 1.123\,12$
$V_{LP}^{(4,\infty)} = 1.513\,19$	$V_{LP}^{(5,\infty)} = 1.905\,33$	$V_{LP}^{(6,\infty)} = 2.297\,05$
$V_{LP}^{(7,\infty)} = 2.688\,38$	$V_{LP}^{(8,\infty)} = 3.079\,54$	$V_{LP}^{(9,\infty)} = 3.470\,63$
$V_{LP}^{(10,\infty)} = 3.861\,71$	$V_{LP}^{(15,\infty)} = 5.817\,04$	$V_{LP}^{(20,\infty)} = 7.772\,38$
$V_{LP}^{(25,\infty)} = 9.727\,72$	$V_{LP}^{(30,\infty)} = 11.683\,07$	$V_{LP}^{(50,\infty)} = 19.504\,37$
$V_{LP}^{(100,\infty)} = 39.057\,72$		

We observe also that, for any m fixed, the energy decreases when n increases. This can be understood using the usual picture since, modifying only one wavefunction, the overlap between the two decreases.

Finally we observe again that the wavelet wavefunctions appear to be better localized than the oscillator ones. This is deduced, this time, since the best localized functions correspond to the maximum in energy. In fact, we have $V_{HO}^{0,1} = 0.918\,73$, $V_{HO}^{0,2} = 0.390\,19$ and $V_{HO}^{1,2} = 0.110\,41$. We see that the maximum of these values corresponds to the most localized wavefunction and, however, is much less than the results one obtains using wavelets.

We conclude the analysis of these results again with the conviction that wavelets can have a strong utility in the problem of finding the ground state of the FQHE.

Acknowledgment

Appendix. Other possible models

In this appendix we want briefly to discuss the main features of other physical models which could be treated with analogous techniques. We only discuss the situation in two spatial dimensions.

The essential ingredient in defining the model is the single-electron Hamiltonian H_0 in (2.1). This operator must satisfy certain constraints. In fact it must be such that a canonical transformation, generalizing the one in (2.3), which transforms the original H_0 in x, y, p_x and p_y into a Hamiltonian depending only on a couple of conjugate variables exists. The most general linear transformation is the following:

$$\tilde{x}_i = \sum_j (a_{ij} x_j + b_{ij} p_j)$$
$$\tilde{p}_i = \sum_j (c_{ij} x_j + d_{ij} p_j)$$

where x_j and p_j are the original canonically conjugate variables and $\tilde{x}_i$ and $\tilde{p}_i$ are the new ones. In [8] this kind of transformation is discussed and, in particular, it is shown how the wavefunction is transformed under this change of variables. In this paper, we have applied the results in [8] only to the Hamiltonian in (2.2). Similar changes of variables can also be applied to other Hamiltonians, such as the following ones:

$$H_1 = \tfrac{1}{2}(p_x^2 + x^2) + \tfrac{1}{2} y^2 + xy$$
$$H_2 = \tfrac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2 + 2xy + 2yp_x + (p_x x + x p_x))$$
$$H_3 = \tfrac{1}{2}(p_x - y/2)^2 + \tfrac{1}{2}(p_y + x/2)^2.$$

In particular, the last one is the Hamiltonian of the FQHE. All these Hamiltonians can be transformed into the Hamiltonian of a harmonic oscillator with a suitable canonical transformation. Of course the link between the wavefunctions in configuration space and in the variables (Q, Q') is different depending on the coefficients a_{ij}, b_{ij}, c_{ij} and d_{ij} and it is often not so easy as in formula (2.5).

The analysis of the FQHE will be discussed in a future paper.

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